

BHARGAVA'S COMPOSITION LAW AND WALDSPURGER'S CENTRAL VALUE THEOREM

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ABSTRACT. We reprove a Waldspurger's formula which relates the toric periods and the central values of L-functions of GL_2 . Our technique, different from the original theta-correspondence approach and the more recent relative trace formula, relies on the exploit of distributions defined on prehomogeneous vector spaces.

1. OVERVIEW

1.1. Introduction. The Waldspurger's theorem on central value of GL_2 L-function is one of the most celebrated results in number theory. One of its applications is the study of periods of automorphic forms. For example, in [GGP12], Gan, Gross and Prasad propose a conjecture on the non-vanishing of period integrals of certain automorphic forms of unitary groups in terms of the special value of L -function; recently Ichino and Ikeda [II10] formulate an analogous conjecture for orthogonal groups.

To the author's knowledge, there are three proofs to the theorem to date. The original one is due to Waldspurger [Wal85], and uses the theta correspondence. More precisely, Waldspurger relates the periods of cusp forms of GL_2 over a non-split torus, via the Shimizu lifting, to the integral representation of the expected L-function. For a more recent account of this method, the reader can consult [YZZ13]. Another approach is due to Jacquet [Jac86] using the relative trace formula. It should be mentioned that Sakellaridis [Sak14] recently provides another proof, still based on the relative formula, which fits into the Langlands' Beyond Endoscopy program.

The purpose of this paper is to propose another proof to Waldspurger's theorem: the key point is to exploit the distributions on certain prehomogeneous vector spaces. The argument is completely independent of any known approach, it might provide the first example along this method to explore the higher-rank cases of the Gross-Prasad conjectures.

1.2. Statement of the result. We will state the theorem for the simplest case, because this is the version we prove in the present paper; but in fact, the general case of the theorem can also be dealt with in an analogous way. We hope to complete the argument of the general case in sequel.

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Let f be an $\mathrm{SL}_2(\mathbb{Z})$ Maass cusp form of weight 0, and π the corresponding automorphic representation of GL_2 with $f \in \pi$. Define the Peterson inner product by

$$\langle \phi, \varphi \rangle = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})} \phi(g) \varphi(g) dg, \quad \phi \in \pi, \varphi \in \pi^\vee.$$

Throughout the paper, we fix an odd fundamental discriminant D . Denote by $\Lambda(D)$ the set of Heegner points of the quadratic extension $E = \mathbb{Q}(\sqrt{D})$ over \mathbb{Q} . Let $\chi_D(\cdot)$ be the quadratic character associated to E/\mathbb{Q} . Then our main theorem is

Theorem 1.1. *Let S be the set of finite places where E is ramified, and denote by π_E the base change from the cuspidal representation π , then we have*

$$(1.2) \quad \frac{|\sum_{z \in \Lambda(D)}^* f(z)|^2}{\|f\|^2} = \frac{h(D)}{8\pi} \frac{\zeta(2)L(1/2, \pi_E)}{L(1, \pi, \mathrm{Ad})L(1, \chi_D)} \prod_{v \in S} P_v,$$

where \sum^* denotes the sum weighted by one over the order of a stabilizer group, and $h(D)$ is the class number of the quadratic field E .

Moreover, the explicit expression for P_v can be obtained.

1.3. Idea of the proof. The method of the proof can be view as a generalization of the Shintani's zeta functions to the higher rank automorphic forms. The Shintani's zeta functions associated to prehomogeneous vector spaces (PVS) was originally defined in [SS74]. Studying the functional equations of these zeta functions, Shintani in [Shi75], [Shi72] obtained the average value of class number of binary quadratic forms and binary cubic forms respectively. To establish the functional equations, one apply the Poisson summation formula and show that the explicit poles and residues are encoded in the orbit integrals associated to singular orbits of PVS. On the other hand, the adelic version of Shintani's functions has shown relevant to the Hecke L-functions in a few cases ([Wri85], [DW86]). However, the adelization of Shintani's zeta function become more difficult to treat in general cases.

The first step of our proof is to realize the period integrals of an automorphic form as the distribution associated to a PVS. This step becomes easy due to M. Bhargava's work on the extension of Gauss's composition law. What matter here is the study of the integral orbits of various PVS, especially the PVS of $2 \times 2 \times 2$ cubes. Forgetting the group structure, Bhargava's composition law states that the integral orbits are in one to one correspondent to the pairs of (strict) class groups of binary quadratic forms. This motivates us to define an integral on the set of $2 \times 2 \times 2$ cubes and to show the integral breaks up into the product of the sum of automorphic form valued at the Heegner points.

The next step is to relate this integral to the special value of L-function of the automorphic form. The key point here is to study a certain parabolic

group action on the skew-symmetrization of $2 \times 2 \times 2$ cubes, i.e., the PVS of pairs of quaternary alternating 2-forms. The structure of integral orbits of this PVS is also studied by Bhargava and shown to be in one to one correspondent to the (strict) class groups of binary quadratic forms. Then we also define a distribution on this PVS associated to the parabolic group. Next we apply the Poisson summation formula to show there is a pole of the distribution and the residue gives rise to the distribution on the PVS of $2 \times 2 \times 2$ cubes defined earlier. In some sense, the $2 \times 2 \times 2$ cubes can be viewed as the singular set of PVS of pairs of quaternary alternating 2-forms. This is made clear if we generalize the PVS of a reductive group to its parabolic subgroups.

The last step is to study the adelic version of the distribution on PVS of pairs of quaternary alternating 2-forms. The reason for this consideration is that there is only one rational orbit of this PVS; moreover, the rational stabilizer group of a representative is the unipotent subgroup of GL_2 and the adelic stabilizer group is a GL_2 . This means integrating along the rational stabilizer group, after the Selberg-Rankin method, becomes an Euler product; furthermore, integrating the adelic stabilizer group gives rise to the matrix coefficients of the automorphic representation. After careful analyzing the local orbit integrals at unramified places via the local orbits counting, we show each of these local integrals is related to the central value of base change L-function of the automorphic form.

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2. PVS OF $2 \times 2 \times 2$ MATRICES

In a seminal series of papers ([Bha04a], [Bha04b], [Bha04c], [Bha08]), M. Bhargava has extended Gauss's composition law for binary quadratic forms to far more general situations. The key step in his extension is the investigation of the integral orbits of a group over \mathbb{Z} acting on a lattice in a prehomogeneous vector space. One of Bhargava's achievements is the determination of the corresponding integral orbits, i.e. the determination of the $\mathrm{SL}_2(\mathbb{Z})^3$ -orbits on $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$. In particular, he discovered that in this case, the generic integral orbits are in bijection with isomorphism classes of tuples (A, I_1, I_2, I_3) where

- (a) A is an order in an étale quadratic \mathbb{Q} -algebra;
- (b) I_1, I_2 and I_3 are elements in the narrow class group of A such that $I_1 \cdot I_2 \cdot I_3 = 1$.

In fact, Bhargava's cube arises naturally in the structure theory of the linear algebraic group of type D_4 . More precisely, let G be a simply connected Chevalley group of this type. It has a maximal parabolic subgroup $P = M \cdot N$ whose Levi factor M has a derived group $M_{\mathrm{der}}(F) \cong \mathrm{SL}_2(F)^3$

and whose unipotent radical N is a Heisenberg group. The adjoint action of $M_{\text{der}}(F)$ on $V = N(F)/[N(F), N(F)]$ is isomorphic to Bhargava's cube.

It is noted that the spaces of forms studied in Bhargava's papers were considered as special examples in the fundamental work of M. Sato and T. Kimura ([SK77]), where the authors completely classified the prehomogeneous vector spaces over \mathbb{C} . Over other fields, such as the rational number field, these spaces were more recently studied in the work of D. Wright and A. Yukie ([WY92]).

2.1. PVS of $2 \times 2 \times 2$ cubes as the spherical variety. Let G be a connected complex Lie group, usually G is a complexification of a real Lie group. A prehomogeneous vector space (PVS) V of G , denoted by (G, V) , is a complex finite dimensional vector space V together with a holomorphic representation of G , such that G has an open dense orbit in V . Let P be a complex polynomial function on V . We call it a relative invariant of G if $P(gv) = \chi(g)P(v)$ for some rational character χ of G .

A spherical variety for a reductive group G over a field k is a normal variety together with a G -action, such that the Borel subgroup of G has a dense orbit. In this section we consider the certain Borel subgroup action on a PVS, and show it again a PVS thus a spherical variety. The main purpose is to show the relative invariants of the Borel subgroup action completely determine the integral orbits.

Let $V(\mathbb{Z})$ be the set of $2 \times 2 \times 2$ integral matrices. For each element $A \in V(\mathbb{Z})$, there are three ways to form pairs of matrices by taking the opposite sides out of 6 sides. Denote them by

$$\begin{aligned} M_A^1 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}; N_A^1 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \\ M_A^2 &= \begin{pmatrix} a & e \\ c & g \end{pmatrix}; N_A^2 = \begin{pmatrix} b & f \\ d & h \end{pmatrix}, \\ M_A^3 &= \begin{pmatrix} a & e \\ b & f \end{pmatrix}; N_A^3 = \begin{pmatrix} c & g \\ d & h \end{pmatrix}. \end{aligned}$$

For each pair (M_A^i, N_A^i) we can associate to it a binary quadratic form by taking

$$Q_A^i(u, v) = -\det(M_A^i u - N_A^i v).$$

Explicitly for A as above,

$$\begin{aligned} -Q_A^1(u, v) &= u^2(ad - bc) + uv(-ah + bg + cf - de) + v^2(eh - fg), \\ -Q_A^2(u, v) &= u^2(ag - ce) + uv(-ah - bg + cf + de) + v^2(bh - df), \\ -Q_A^3(u, v) &= u^2(af - be) + uv(-ah + bg - cf + de) + v^2(ch - dg). \end{aligned}$$

Following Bhargava [Bha04a], we call A projective if the associated binary quadratic forms are all primitive. The action of group $G(\mathbb{Z}) = \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})$ on $V(\mathbb{Z})$ is defined by taking the g_i in (g_1, g_2, g_3) acts on the matrix pair (M_A^i, N_A^i) . It is easy to check that the actions of the

three components commute with each other, thereby giving an action of the product group. For example, if $g_1 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, then it acts on the pair (M_A^1, N_A^1) by

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} M_A^1 \\ N_A^1 \end{pmatrix}.$$

The action extends to the complex group $G(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ on the complexified vector space $V(\mathbb{C})$. Now we consider the Borel subgroup $B'_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$ consisting of the lower-triangular matrices. The action of the subgroup $B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ is induced from full group action and has three relative invariants, explicitly for $A \in V(\mathbb{C})$, given as follows

$$\begin{aligned} D(A) &= \mathrm{disc}(A) = (-ah + bg + cf - de)^2 - 4(ad - bc)(eh - fg), \\ m(A) &= -\det(M_A^1) = -(ad - bc), \\ n(A) &= -\det(M_A^2) = -(ag - ce). \end{aligned}$$

The corresponding rational character are

$$\begin{aligned} \chi_1(g) &= \det(b_1)^2 \det(b_2)^2 \det(g_3)^2, \\ \chi_2(g) &= r_1^2 \det(b_2) \det(g_3), \\ \chi_3(g) &= \det(b_1) r_2^2 \det(g_3), \end{aligned}$$

$$\text{for } g = (b_1, b_2, g_3) = \left(\begin{pmatrix} r_1 & 0 \\ u_1 & s_1 \end{pmatrix}, \begin{pmatrix} r_2 & 0 \\ u_2 & s_2 \end{pmatrix}, g_3 \right).$$

Furthermore, one can easily show that

Proposition 2.1. *The pair $(B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}), V(\mathbb{C}))$ is a prehomogeneous vector space.*

The immediate corollary is that $(G(\mathbb{C}), V(\mathbb{C}))$ is again a PVS which is called the D_4 type studied in [WY92].

2.1.1. Shintani's double Dirichlet series. Our consideration of the Borel subgroup action on a PVS is inspired by the Shintani's work [Shi75], where the author studies the zeta function associated to PVS of binary quadratic forms and hence obtains the average value of the size of class group of quadratic fields. We recall briefly here the Shintani's zeta function.

Let $U(\mathbb{C}) = \{Q(u, v) = au^2 + buv + cv^2 \mid (a, b, c) \in \mathbb{C}^3\}$ be the complex vector space of binary quadratic forms. The representation ρ of the group $\mathrm{GL}_2(\mathbb{C})$ is as follows

$$\rho(g)(Q)(u, v) = Q(au + cv, bu + dv)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It is easy to see that $(B'_2(\mathbb{C}), \rho, U(\mathbb{C}))$ is a PVS and has two relative invariants, namely, the discriminant $\mathrm{disc}(Q) = b^2 - 4ac$ of the

quadratic form and $a = Q(1, 0)$. These two invariants freely generate the ring of relative invariants.

Let $B'_2(\mathbb{Z})$ be the Borel subgroup $B'_2(\mathbb{C}) \cap \mathrm{SL}_2(\mathbb{Z})$. The Shintani's zeta function associated to the PVS $(B'_2(\mathbb{C}), \rho, U(\mathbb{C}))$ is defined to be

$$\begin{aligned} Z(s, w) &= \sum_{Q \in B'_2(\mathbb{Z}) \setminus U^{ss}(\mathbb{Z})} \frac{1}{|Q(1, 0)|^s |\mathrm{disc}(Q)|^w} \\ &= \sum_{a \neq 0} \frac{1}{|a|^s} \sum_{\substack{0 \leq b \leq 2a-1 \\ c: b^2 - 4ac \neq 0}} \frac{1}{|b^2 - 4ac|^w}. \end{aligned}$$

where $U^{ss}(\mathbb{Z})$ is the semi-stable subset of (not necessarily primitive) integral quadratic forms with both $\mathrm{disc}(Q), Q(1, 0) \neq 0$. Alternatively, one can express the Shintani's zeta function as

$$\sum_{Q \in \mathrm{SL}_2(\mathbb{Z}) \setminus U^{ss}(\mathbb{Z})} \frac{1}{|\mathrm{disc}(Q)|^w} \sum_{\gamma \in B'_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{Z}) / \mathrm{stab}_Q} \frac{1}{|\gamma \cdot Q(1, 0)|^s}.$$

If we denote by $A(d, a)$ the number of solutions to the quadratic congruence equation $x^2 = d \pmod{a}$. Then Shintani's zeta function can be written as

$$Z(s, w) = \xi_1(s, w) + \xi_2(s, w),$$

where $\xi_i(s, w) = \sum_{a, d > 0} \frac{A((-1)^{i-1} d, 4a)}{a^s d^w}$.

The study of multi-variable Dirichlet series receives extensive attentions in recent years, especially on the development of Weyl group multiple Dirichlet series (WMDS). Those who are interested in WMDS can consult the expository paper [BBC⁺06]. Though the study of relation between WMDS and Shintani's zeta function associated to the PVS is not our purpose in this paper, we show that $Z(s, w)$ is a A_2 Weyl group multiple Dirichlet series. The technique used in the proof will be needed again in the last section to obtain the local L-function in Waldspurger's theorem. We start with the definition of a A_2 WMDS.

Weyl group multiple Dirichlet series are a class of multiple Dirichlet series coming from Eisenstein series on metaplectic groups. The simplest example is the quadratic A_2 Weyl group double Dirichlet series (see for instance [CG07], [CG10]), it is defined as

$$Z_{A_2}(s, w) = \sum_{\substack{m > 0, \\ D \text{ odd discriminant}}} \frac{\chi_D(\hat{m})}{m^s |D|^w} a(D, m),$$

where \hat{m} is the factor of m that is prime to the square-free part of D and χ_D is the quadratic character associated to the field extension $\mathbb{Q}(\sqrt{D})$ of \mathbb{Q} . More interesting to us is the multiplicative factor $a(D, m)$, which is defined by

$$a(D, m) = \prod_{p^k || D, p^l || m} a(p^k, p^l)$$

and

$$a(p^k, p^l) = \begin{cases} \min(p^{k/2}, p^{l/2}) & \text{if } \min(k, l) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the relation between Shintani's zeta function $Z(s, w)$ and $Z_{A_2}(s, w)$ is implied by

Proposition 2.2. *Fix D an odd discriminant. Then*

$$\sum_{m>0} \frac{A(D, 4m)}{m^s} = 2 \frac{\zeta(s)}{\zeta(2s)} \sum_{m>0} \frac{\chi_D(\hat{m}) a(D, m)}{m^s}.$$

Proof. Analogous to the p -part formula of $a(p^k, p^l)$, we will show that for an odd prime p

$$A(p^k, p^l) = \begin{cases} 2a(p^k, p^l) & \text{if } k < l, \\ p^{\lfloor l/2 \rfloor} & \text{otherwise.} \end{cases}$$

First consider the case when $p \neq 2$. If $k < l$ and k is an odd integer, then the congruence equation $x^2 = p^k \pmod{p^l}$ reduces to the equation of $x^2 = p \pmod{p^i}$ for some power i of p and there is no solution to it; while when k is an even integer, the congruence equation $x^2 = p^k \pmod{p^l}$ reduces to the equation of $x^2 = 1 \pmod{p^i}$ and there are two solutions to it. In both case, the number of solutions are both equal to the value of $2a(p^k, p^l)$ by its definition.

If on the other hand $k \geq l$, then the set of solutions to the congruence equation $x^2 = p^k \pmod{p^l}$ is the set of multipliers of $p^{\lceil \frac{l}{2} \rceil}$, so the number of distinct solutions mod p^l is $p^{\lfloor \frac{l}{2} \rfloor}$.

Therefore, for odd prime p ,

$$A(p^k, p^l) = \chi_{p^k}(\hat{p}^l) a(p^k, p^l) + \chi_{p^k}(\hat{p}^{l-1}) a(p^k, p^{l-1}) = a(p^k, p^l) + a(p^k, p^{l-1}),$$

where we set the term $a(p^k, p^{l-1})$ equal to 0 when $l = 0$.

Next, using Hensel's lemma, an integer d relatively prime to an odd prime p is a quadratic residue modulo any power of p if and only if it is a quadratic residue modulo p . In fact, if an integer d is prime to the odd prime p , as

$$A(d, p^l) = 2 \iff \chi_d(p) = 1 \iff A(d, p) = 2,$$

so

$$A(d, p^l) = \chi_d(p^l) + \chi_d(p^{l-1}).$$

By the prime power modulus theory [Gau66], if the modulus is p^l , then $p^k d$ is a

$$\begin{cases} \text{quadratic residue modulo } p^l & \text{if } k \geq l, \\ \text{non-quadratic residue modulo } p^l & \text{if } k < l \text{ is odd,} \\ \text{quadratic residue modulo } p^l & \text{if } k < l \text{ is even and } d \text{ is a quadratic residue,} \\ \text{non-quadratic residue modulo } p^l & \text{if } k < l \text{ is even and otherwise.} \end{cases}$$

Therefore, for an odd integer d prime to $p \neq 2$, we have

$$A(dp^k, p^l) = \begin{cases} 0 & \chi_d(p^l) = -1 \text{ and } k < l \text{ even,} \\ A(p^k, p^l) & \text{otherwise.} \end{cases}$$

In the former case, we have:

$$(1) \quad A(dp^k, p^l) = 0 = \chi_{dp^k}(\hat{p}^l)a(dp^k, p^l) + \chi_{dp^k}(\hat{p}^{l-1})a(dp^k, p^{l-1}).$$

In the latter case, we also have:

$$(2) \quad A(dp^k, p^l) = A(p^k, p^l) = \chi_{dp^k}(\hat{p}^l)a(dp^k, p^l) + \chi_{dp^k}(\hat{p}^{l-1})a(dp^k, p^{l-1}).$$

Now let D be an arbitrary odd discriminant. Given an prime integer p , write $D = D_0 p^k$, where D_0 is prime to p . Then from the equality (1) and (2) with d replaced by D_0 , it follows that

$$\sum_{l=0}^{\infty} A(D, p^l) p^{-ls} = (1 - p^{-2s})(1 - p^{-s})^{-1} \sum_{l=0}^{\infty} \chi_D(\hat{p}^l) a(D, p^l) p^{-ls}.$$

For $p = 2$, we define $\tilde{P}_2(D, s)$ by equating

$$\begin{aligned} \sum_{l=0}^{\infty} \frac{A(D, 2^{l+2})}{2^{ls}} &= \tilde{P}_2(D, s)(1 - 2^{-2s})(1 - 2^{-s})^{-1} \sum_{l=0}^{\infty} \frac{\chi_D(2^l)}{2^{ls}} \\ &= \tilde{P}_2(D, s)(1 - 2^{-2s})(1 - 2^{-s})^{-1} \sum_{l=0}^{\infty} \frac{\chi_D(2^l) a(D, 2^l)}{2^{ls}}. \end{aligned}$$

By the multiplicative property of $A(D, \cdot)$ and that of $\chi_D(\cdot)a(D, \cdot)$,

$$\sum_{m>0} \frac{A(D, 4m)}{m^s} = \tilde{P}_2(D, s) \zeta(2s)^{-1} \zeta(s) \sum_{m>0} \frac{\chi_D(\hat{m}) a(D, m)}{m^s}.$$

It remains to compute \tilde{P}_2 . This is obtained by the next lemma.

Lemma 2.3. *Let D be an odd discriminant. With $\tilde{P}_2(D, s)$ defined in the last proposition, then*

$$\tilde{P}_2(D, s) = 2.$$

Proof. Write

$$\sum_{l=0}^{\infty} \frac{A(D, 2^{l+2})}{2^{ls}} = A(D, 4) + \sum_{l=1}^{\infty} \frac{A(D, 2^{l+2})}{2^{ls}},$$

and note that

$$A(D, 4) = \begin{cases} 2 & D \equiv 1 \text{ or } 5 \pmod{8}, \\ 0 & \text{otherwise.} \end{cases}$$

To simplify the second term, note that if D is an odd integer and $m = 8, 16$, or some higher power of 2, then D is a quadratic residue modulo m if and

only if $D \equiv 1 \pmod{8}$, therefore for $l \geq 1$,

$$A(D, 2^{l+2}) = \begin{cases} 4 & D \equiv 1 \pmod{8}, \\ 0 & \text{otherwise.} \end{cases}$$

Also note that

$$\chi_D(2) = \begin{cases} 1 & D \equiv 1 \pmod{8}, \\ -1 & D \equiv 5 \pmod{8}, \\ 0 & \text{otherwise.} \end{cases}$$

Then direct computation gives the results. \square

This finishes the proof of the proposition. \square

2.1.2. Reduction theory. In this section, we study integral orbits of the PVS $(B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}), V(\mathbb{C}))$. We are going to show that the integral orbit in this PVS is completely determined by its three relative invariants.

Lemma 2.4. *Let D , m and n be non-zero integers. For each solution (x, y) to the congruence equations*

$$x^2 \equiv D \pmod{4m} \text{ for } 0 \leq x \leq 2|m| - 1,$$

$$y^2 \equiv D \pmod{4n} \text{ for } 0 \leq y \leq 2|n| - 1,$$

there exists a $2 \times 2 \times 2$ integer cube A such that

$$\mathrm{disc}(A) = D,$$

$$Q_A^1(u, v) = mu^2 + xuv + sv^2,$$

$$Q_A^2(u, v) = nu^2 + yuv + tv^2.$$

Moreover, the required $2 \times 2 \times 2$ integer cube A can be chosen such that in the top side M_A^3 of A

$$a = 0 \text{ and } g.c.d.(b, e, f) = 1.$$

Proof. If there are solutions to the congruence equations, we have

$$D = x^2 - 4ms = y^2 - 4nt$$

for some integers s and t . It implies that D is congruent to 0 or 1 $\pmod{4}$, and the integers x, y have the same parity. Take

$$c = |g.c.d.(m, n, \frac{x+y}{2})|,$$

from

$$(1) \quad \frac{x-y}{2} \cdot \frac{x+y}{2} = ms - nt,$$

it follows that

$$(2) \quad g.c.d.(\frac{m}{c}, \frac{n}{c}) \mid \frac{x-y}{2}.$$

Set

$$b = \frac{m}{c}, \quad e = \frac{n}{c}, \quad \text{and } f = -\frac{x+y}{2c},$$

then

$$g.c.d.(b, e, f) = 1,$$

and (1) can be written as

$$(3) \quad \frac{x-y}{2} \cdot (-f) = bs - et.$$

We claim that there is an integer h , such that

$$s + eh \equiv 0 \pmod{f},$$

$$t + bh \equiv 0 \pmod{f}.$$

This can be proved as follows: First if $f = 0$, then $bs = et$. As in this case $g.c.d.(b, e) = 1$, we conclude that there exists such an integer h such that $s = -eh$ and $t = -bh$. If $f \neq 0$, for any prime divisor p of f , we have

$$p \mid bs - et \text{ and } g.c.d.(b, e, f) = 1,$$

it follows that there is a unique solution $(\text{mod } p)$ to the congruences

$$s + eh \equiv 0 \pmod{p},$$

$$t + bh \equiv 0 \pmod{p}.$$

Using the Chinese remainder theorem, we conclude that there are integers d and g such that

$$s = gf - eh,$$

$$t = df - bh.$$

It follows that

$$bs - et = (bg - de) \cdot f,$$

combined with (3), we have

$$\frac{x-y}{2} = de - bg.$$

If in the case of $f = 0$, from (2), we know that there always exist integers d and g such that the above equation holds.

Now we define a $2 \times 2 \times 2$ integer cube A by

$$(M_A^1, M_A^2) = \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right).$$

Then the two associated binary quadratic forms are

$$Q_A^1(u, v) = bcu^2 - (bg + cf - de)uv - (eh - fg)v^2 = mu^2 + xuv + sv^2,$$

$$Q_A^2(u, v) = ceu^2 - (-bg + cf + de)uv - (bh - df)v^2 = nu^2 + yuv + tv^2.$$

So A is the cube required. In particular, we have shown that $g.c.d.(b, e, f) = 1$. \square

We next want to show that under the assumption that D is square-free, the data (D, m, n, x, y) uniquely determines a $B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})$ -orbit.

Proposition 2.5. *Let D be a non-zero square-free integer and m, n be non-zero integers. Each solution to the congruence equations*

$$\begin{aligned} x^2 &\equiv D \pmod{4m} \text{ for } 0 \leq x \leq 2|m| - 1, \\ y^2 &\equiv D \pmod{4n} \text{ for } 0 \leq y \leq 2|n| - 1, \end{aligned}$$

determines uniquely an integral orbit represented by A such that

$$\begin{aligned} \text{disc}(A) &= D, \\ Q_A^1(u, v) &= mu^2 + xuv + sv^2, \\ Q_A^2(u, v) &= nu^2 + yuv + ty^2. \end{aligned}$$

Proof. From the last lemma, we know that there always exists such a $2 \times 2 \times 2$ integral matrix A satisfying the congruence equations. We want to show that the integral orbit it represent in the PVS $(B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}), V(\mathbb{C}))$ is uniquely determined by the data (d, m, n, x, y) .

Denote A by the pair of matrices

$$A = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right).$$

Under the action of the subgroup $1 \times 1 \times \text{SL}_2(\mathbb{Z})$, $a(A)$ can be made equal to 0, so we assume it. We then show that the (c, b, e, f) is uniquely determined up to the sign of c . Then we have equations

$$\begin{aligned} -bg - cf + de &= x, \\ bg - cf - de &= y, \\ bc &= m, \\ ce &= n. \end{aligned}$$

after adding the first two equations,

$$\begin{aligned} cf &= -(x + y)/2, \\ bc &= m, \\ ce &= n. \end{aligned}$$

Therefore we have

$$c \times g.c.d.(b, e, f) = g.c.d.(m, n, (x + y)/2).$$

As

$$D = (bg - ah - ed)^2 + 4bc(eh - fg),$$

and D is square-free, it implies that $g.c.d.(b, e, f) = 1$. Therefore $c = |g.c.d.(m, n, (x + y)/2)|$ as we can transform c to be positive.

We next show that for two such $2 \times 2 \times 2$ integral matrices with $a = 0$ they are equivalent in the sense one is transformed to another by the action of $1 \times 1 \times B'_2(\mathbb{Z})$. To prove this statement we again require D to be square-free.

As both have the same associated binary quadratic forms:

$$Q_A^1(u, v) = mu^2 + xuv + sv^2 = bcu^2 - (bg + cf - ed)uv - (eh - fg)v^2,$$

$$Q_A^2(u, v) = nu^2 + yuv + tv^2 = ceu^2 - (-bg + cf + de)uv - (bh - df)v^2,$$

we can set up the following equations by comparing the coefficients:

$$x(A_1) = x(A_2) : bg_1 + cf - ed_1 = bg_2 + cf - ed_2,$$

$$y(A_1) = y(A_2) : ed_1 + cf - bg_1 = ed_2 + cf - bg_2,$$

$$s(A_1) = s(A_2) : eh_1 - fg_1 = eh_2 - fg_2,$$

$$t(A_1) = t(A_2) : bh_1 - d_1f = bh_2 - d_2f.$$

Subtracting the right side from the left side on each equation, we have:

$$b\Delta(g) - e\Delta(d) = 0,$$

$$e\Delta(h) - f\Delta(g) = 0,$$

$$b\Delta(h) - f\Delta(d) = 0,$$

where we use notation $\Delta(g) = g_1 - g_2$. Then we have solutions to the above equation system:

$$\Delta(d); \Delta(g) = \Delta(d)e/b; \Delta(h) = \Delta(d)f/b.$$

Under the assumption of D square-free, we have shown that $\text{g.c.d.}(b, e, f) = 1$. Therefore $b|\Delta(d)$, i.e., the solution has form

$$\Delta(d) = bk; \Delta(g) = ek; \Delta(h) = fk.$$

This implies that we can make A_1 equivalent to A_2 by the action of $1 \times 1 \times B_2'(\mathbb{Z})$. \square

Now we turn to the general case without assuming square-free discriminant. For this, we denote by $B(D, m, n)$ the number of integral orbits $[A]$ in $(B_2'(\mathbb{C}) \times B_2'(\mathbb{C}) \times \text{GL}_2(\mathbb{C}), V(\mathbb{C}))$ with $\text{disc}(A) = D, Q_A^1(1, 0) = m, Q_A^1(1, 0) = n$. Then

Proposition 2.6. *Let m and n be non-zero integers, and $D = D_0 D_1^2$ where D_0 is square-free. We have*

$$B(D, m, n) = \frac{1}{4} \sum_{d|D_1} b\left(\frac{D}{d^2}, \frac{m}{d}, \frac{n}{d}\right),$$

where

$$b\left(\frac{D}{d^2}, \frac{m}{d}, \frac{n}{d}\right) = \begin{cases} d \cdot A\left(\frac{D}{d^2}, \frac{4m}{d}\right) \cdot A\left(\frac{D}{d^2}, \frac{4n}{d}\right) & \text{if } d \text{ divides } \text{g.c.d.}(D_1, m, n), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the existence lemma, there is a $2 \times 2 \times 2$ integral matrix A satisfying

$$a = 0 \text{ and } \text{g.c.d.}(b, e, f) = 1,$$

and such that

$$\text{disc}(A) = D, Q_A^1(1, 0) = m \text{ and } Q_A^1(1, 0) = n.$$

The condition $g.c.d.(b, e, f) = 1$ implies that the matrix A determines a unique integral orbit.

If $d | g.c.d.(D_1, m, n)$, we write $m' = \frac{m}{d}$, $n' = \frac{n}{d}$. Applying the existence lemma again to the non-zero integers $\frac{D}{d^2}$, $\frac{m}{d}$, and $\frac{n}{d}$, we can find an $2 \times 2 \times 2$ integral matrix A' satisfying

$$a' = 0 \text{ and } g.c.d.(b', e', f') = 1,$$

and such that

$$\text{disc}(A') = \frac{D}{d^2}, \quad Q_A^1(1, 0) = \frac{m}{d} \text{ and } Q_A^2(1, 0) = \frac{n}{d}.$$

The condition $g.c.d.(b', e', f') = 1$ implies that A' determines a unique integral orbit with discriminant D/d^2 .

For each g in $\{1 \times 1 \times \begin{pmatrix} 1 & 0 \\ j & d \end{pmatrix} : 0 \leq j \leq d-1\}$, the integral cube $g \cdot A'$ represents an integral orbit with discriminant D . \square

As an immediately corollary, we have

Corollary 2.7. *If D is a fundamental discriminant, then*

$$B(D, m, n) = \frac{1}{4} A(D, 4m) A(D, 4n).$$

3. PVS OF PAIRS OF QUATERNARY ALTERNATING 2-FORMS

In [Bha04a], Bhargava studies 5 other PVS obtain from the D_4 type via various operations such as symmetrizing, anti-symmetrizing, castling, etc. For example, the if one impose the conditions skew symmetry on the set of cubes, it will produce a new set consisting of pairs of quaternary alternating 2-forms via the fusion process. Let $\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4$ be the space of paris of quaternary alternating 2-forms. Denote its element $F = (M_F, N_F)$ by

$$\left(\begin{pmatrix} 0 & r_1 & a & b \\ -r_1 & 0 & c & d \\ -a & -c & 0 & l_1 \\ -b & -d & -l_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_2 & e & f \\ -r_2 & 0 & g & h \\ -e & -g & 0 & l_2 \\ -f & -h & -l_2 & 0 \end{pmatrix} \right).$$

The group $\text{SL}_2(\mathbb{Z}) \times \text{SL}_4(\mathbb{Z})$ acts naturally on it. Explicitly, the action of $\text{SL}_2(\mathbb{Z}) \times \text{SL}_4(\mathbb{Z})$ is given as follows: an element $(g_1, g_2) \in \text{SL}_2(\mathbb{Z}) \times \text{SL}_4(\mathbb{Z})$ acts by sending the pair $F = (M_F, N_F)$ to:

$$(g_1, g_2) \cdot (M_F, N_F) = (s \cdot g_2 M_F g_2^t + t \cdot g_2 N_F g_2^t, u \cdot g_2 M_F g_2^t + v \cdot g_2 N_F g_2^t),$$

where $g_1 = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$. Then Bhargava's fusion process is a \mathbb{Z} -linear mapping:

$$\text{id} \otimes \wedge_{2,2} : \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \otimes \wedge^2(\mathbb{Z}^2 \oplus \mathbb{Z}^2) = \mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4.$$

Explicitly, it is given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \rightarrow \left(\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & e & f \\ 0 & 0 & g & h \\ -e & -g & 0 & 0 \\ -f & -h & 0 & 0 \end{pmatrix} \right).$$

Bhargava further proves the mapping above is surjective on the level of equivalence classes.

To each $F = (M_F, N_F) \in \mathbb{Z}^2 \otimes \mathbb{Z}^4$, we can associate a binary quadratic form Q given by:

$$Q_F(u, v) = -\text{Pfaff}(M_F u - N_F v) = -\sqrt{\text{Det}(M_F u - N_F v)},$$

where the sign of the Pfaff is chosen by taking

$$\text{Pfaff} \left(\begin{pmatrix} & I \\ -I & \end{pmatrix} \right) = 1,$$

or alternatively,

$$\text{Pfaff} \left(\begin{pmatrix} 0 & r & a & b \\ -r & 0 & c & d \\ -a & -c & 0 & l \\ -b & -d & -l & 0 \end{pmatrix} \right) = ad - bc - rl.$$

We now consider the subgroup of $\text{SL}_2(\mathbb{Z}) \times \text{SL}_4(\mathbb{Z})$ defined by $H'(\mathbb{Z}) = B'_2(\mathbb{Z}) \times P'_{2,2}(\mathbb{Z})$, where $P'_{2,2}(\mathbb{Z})$ is the lower triangular minimal parabolic subgroup of $\text{SL}_4(\mathbb{Z})$ that has the shape

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ s & u & * & * \\ t & v & * & * \end{pmatrix}$$

with the upper left 2×2 matrix determinant 1. Let $W(\mathbb{C})$ be the subspace of elements with vanishing r_1 -position. Then the action extends to the complex group $H'(\mathbb{C})$ on complexified vector space $W(\mathbb{C})$. It has three relative invariants, they are

$$\text{disc}(F) = \text{disc}(Q_F),$$

$$P_0(F) = r_2(F),$$

$$P_1(F) = -\text{Pfaff}(M_F).$$

We denote by χ_0, χ_1 the characters of group $H'(\mathbb{C})$ determined by the polynomial invariants P_0, P_1 respectively, i.e.,

$$P_0(gF) = \chi_0(g)P_0(F) \text{ and } P_1(gF) = \chi_1(g)P_1(F).$$

One can see easily that

Proposition 3.1. *The pair $(H'(\mathbb{C}), W(\mathbb{C}))$ is a prehomogeneous vector space.*

In the next section we will define distributions on two PVS, $(G(\mathbb{C}), V(\mathbb{C}))$ and $(H'(\mathbb{C}), W(\mathbb{C}))$ respectively, associated to a Maass cusp form of GL_2 . We will further show that the distribution on $(G(\mathbb{C}), V(\mathbb{C}))$ is roughly the product of periods of the Maass form. The key point there is the Bhargava's composition law on $2 \times 2 \times 2$ integral cubes. To end this section we mention this composition law on $2 \times 2 \times 2$ cubes and further on pairs of quaternary alternating 2-forms.

One of the remarkable results of Bhargava is showing that the set of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ -equivalent classes of projective cubes can be quipped with the group structure. We denote this group by $\mathrm{Cl}(\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2; D)$, and the strict ideal class group of the quadratic ring $R(D)$ by $\mathrm{Cl}^+(R(D))$. Then the composition law on $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ is

$$\mathrm{Cl}(\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2; D) \cong \mathrm{Cl}^+(R(D)) \times \mathrm{Cl}^+(R(D))$$

via the map $[A] \rightarrow ([Q_A^1], [Q_A^2])$. Analogously, the $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_4(\mathbb{Z})$ -equivalent classes of projective element in $\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4$ with discriminant D also possess a group structure. If we denote this group by $\mathrm{CL}(\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4; D)$, the composition law are

$$\mathrm{CL}(\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4; D) \cong \mathrm{Cl}^+(R(D))$$

via the map $[F] \rightarrow [Q_F]$, as well as a surjective group homomorphism

$$\mathrm{Cl}(\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2; D) \rightarrow \mathrm{CL}(\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4; D)$$

via the map $[A] \rightarrow [\mathrm{id} \otimes \wedge_{2,2}(A)]$.

4. PERIODS OF AUTOMORPHIC FORMS OF GL_2

From now on, we fix f a Maass cusp eigenform f with weight 0 and full level group $\mathrm{SL}_2(\mathbb{Z})$. Let π the automorphic representation of GL_2 such that $f \in \pi$. Denote by ν the corresponding eigenvalue of the Laplace operator. In order to define distributions on PVS associated to the Maass form f , we need to consider the right action of groups on their PVS.

Let k be any field. Denote $V(k)$ the vector space of $2 \times 2 \times 2$ matrices. For each $v \in V$, write $v = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right)$. We define the right action of $(g_1, g_2, g_3) \in \mathrm{GL}_2(k) \times \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$ on $V(k)$ is given by

$$(M_v^i, N_v^i) \rightarrow (M_v^i, N_v^i)g_i,$$

where the operation is the matrix multiplication.

Let $k^2 \otimes \wedge^2 k^4$ be the vector subspace of pairs of quaternary alternating 2-forms over k such that $w = \left(\begin{pmatrix} 0 & r_1 & a & b \\ -r_1 & 0 & c & d \\ -a & -c & 0 & l_1 \\ -b & -d & -l_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_2 & e & f \\ -r_2 & 0 & g & h \\ -e & -g & 0 & l_2 \\ -f & -h & -l_2 & 0 \end{pmatrix} \right)$

for $w \in k^2 \otimes \wedge^2 k^4$. We define the right action of $(g_1, g) \in \mathrm{GL}_2(k) \times \mathrm{GL}_4(k)$ is given by

$$(M_w, N_w) \rightarrow (sg^t M_w g + ug^t N_w g, tg^t M_w g + vg^t N_w g),$$

where $g_1 = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$.

4.1. Periods of GL_2 Maass forms. Assume from now on D an odd fundamental discriminant. Let $V_D(\mathbb{Z})$ be the set of integral $2 \times 2 \times 2$ matrices with discriminant D . Denote the group $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Let $\mathrm{GL}_2^+(\mathbb{R})$ be the group with positive determinant and let the Haar measure on $\mathrm{GL}_2^+(\mathbb{R})$ be defined by

$$(\det(g))^{-2} \prod_{1 \leq i, j \leq 2} dg_{ij}.$$

To any Schwartz function ϕ on $V(\mathbb{R})$, we associated a mixed Eisenstein series of $G(\mathbb{R})$ by

$$\Phi(s_1, g) = E^*(s_1, g_1) \sum_{v \in V_D(\mathbb{Z})} \phi(vg),$$

where $s_1 \in \mathbb{C}$ and $E^*(s_1, g_1)$ is the usual normalized Eisenstein series of GL_2 with residue $\frac{1}{2}$ at the simple pole $s_1 = 1$.

Now define a distribution, and call it the Shintani integral, on $(G(\mathbb{R}), V_D(\mathbb{R}))$ by

$$J(s_1, f, \phi) = \int_{G(\mathbb{Z}) \backslash G(\mathbb{R})} \Phi(s_1, g) f(g_2) \bar{f}(g_3) dg.$$

We will fix a Schwartz function on $V(\mathbb{R})$. For $v \in V(\mathbb{R})$, define it by

$$\phi(v) = \exp^{-\pi\tau(a(v)^2 + b(v)^2 + c(v)^2 + d(v)^2)} \cdot \exp^{-\pi\tau(e(v)^2 + f(v)^2 + g(v)^2 + h(v)^2)},$$

where τ is a positive real parameter. Next we compute the residue of $J(s_1, f, \phi)$ at $s_1 = 1$.

Theorem 4.1. *The Shintani integral $J(s_1, f, \phi)$ has a simple pole at $s_1 = 1$ with residue*

$$\begin{aligned} & \mathrm{Res}_{s_1=1} J(s_1, f, \phi) \text{ as } \tau \rightarrow \infty \\ & \sim 2L(\phi) \cdot \left| \sum_{z \in \Lambda_D}^* f(z) \right|^2, \end{aligned}$$

where the distribution L is given by

$$L(\phi) = \int_{v \in G(\mathbb{R})} \phi(vg) dv,$$

where v is any element in $V_D(\mathbb{R})$ and \sum^* denotes the sum weighted by one over the order of the stabilizer.

Proof. The proof, including taking the asymptotic as $\tau \rightarrow \infty$, is inspired by the calculation in [KS93, §3]. Unfolding the sum over representatives of $V_D(\mathbb{Z})$ then taking residue of Eisenstein series, we have

$$\begin{aligned}
& \text{Res}_{s_1=1} J(s_1, f, \phi) \\
&= \text{Res}_{s_1=1} \sum_{v \in V_D(\mathbb{Z})/G(\mathbb{Z})} \int_{G_v(\mathbb{Z}) \backslash G(\mathbb{R})} E^*(s_1, g_1) \phi(vg) f(g_2) \bar{f}(g_3) dg \\
&= \frac{1}{2} \sum_{v \in V_D(\mathbb{Z})/G(\mathbb{Z})} \int_{G_v(\mathbb{Z}) \backslash G(\mathbb{R})} \phi(vg) f(g_2) \bar{f}(g_3) dg \\
&= \frac{1}{2} \sum_{v \in V_D(\mathbb{Z})/G(\mathbb{Z})}^* \int_{G(\mathbb{R})} \phi(vg) f(g_2) \bar{f}(g_3) dg,
\end{aligned}$$

where \sum^* denotes the sum weighted by one over the order of the stabilizer.

Note that $V(\mathbb{R})$ consists of 4 components determined by the sign of $Q_v^1(1, 0)$ and $Q_v^2(1, 0)$ for $v \in V(\mathbb{R})$. As $G(\mathbb{R})$ acts transitively on each component and $\text{SL}_2(\mathbb{R})$ acts transitively on the $\{(a, b, c) : b^2 - 4ac = D; a > 0\}$, for any $v \in V_D(\mathbb{Z})$ chosen in the component $Q_v^1(1, 0), Q_v^2(1, 0) > 0$, we can find an element $h = (h_1, h_2, h_3) \in G(\mathbb{R})$ such that

$$Q_{vh}^i(u_1, u_2) = \frac{\sqrt{|D|}}{2} (u_1^2 + u_2^2)$$

or $i = 1, 2, 3$. Explicitly, if $Q_v^1(u_1, u_2) = mu_1^2 + xu_1u_2 + su_2^2$ and $m > 0$, then h_1 is specified to be

$$h_1 = \begin{pmatrix} 1 & -x/2m \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} (\frac{\sqrt{|D|}}{2m})^{1/2} & 0 \\ 0 & (\frac{\sqrt{|D|}}{2m})^{-1/2} \end{pmatrix}.$$

Then h_1 defines the Heegner point z corresponding to the form $mu^2 + xuv + sv^2$. Put

$$v_0 = \left(\begin{pmatrix} 0 & (\frac{\sqrt{|D|}}{2})^{1/2} \\ (\frac{\sqrt{|D|}}{2})^{1/2} & 0 \end{pmatrix}, \begin{pmatrix} (\frac{\sqrt{|D|}}{2})^{1/2} & 0 \\ 0 & -(\frac{\sqrt{|D|}}{2})^{1/2} \end{pmatrix} \right).$$

Then after changing variables $g_1 \rightarrow h_1 g_1$, $g_2 \rightarrow h_2 g_2$ and $g_3 \rightarrow h_3 g_3$, $\text{Res}_{s_1=1} J(s_1, f, \phi)$ becomes

$$\frac{4}{2} \sum_{(z_i, z_j) \in \Lambda(D) \times \Lambda(D)} \int_{G(\mathbb{R})} \phi(v_0 g) f(h_{i2} g_2) \bar{f}(h_{j3} g_3) dg, .$$

where we apply the Bhargava's law of composition to parametrize the integral orbits in each component of $V_D(\mathbb{Z})$ by the pairs of Heegner points, the 4 in the numerator is because there are 4 components and each contributes to the same sum of orbit integrals.

Now using the Cartan decomposition $\text{SL}_2(\mathbb{R}) = \text{SO}_2(\mathbb{R})A^+\text{SO}_2(\mathbb{R})$, as ϕ is right SO_2 invariant and v_0 has stabilizer group isomorphic to $\text{SO}_2 \times \text{SO}_2$,

we have for each orbit integral

$$\int_{a>1} \int_K \int_K \phi(v_0 k' a k) f(h_{i2} k'_2 a_2 k_2) \bar{f}(h_{j3} k'_3 a_3 k_3) \delta(a) \frac{da}{a} dk' dk,$$

where $\delta(a) = \prod_{i=1}^3 \frac{a_i^2 - a_i^{-2}}{2}$ and $K = \mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R})$. Now

$$\int f(h_{i2} k'_2 a_2 k_2) dk'_2 dk_2$$

is again an eigenfunction of the Laplace operator with the same eigenvalue ν as $f \in \pi$, hence by the uniqueness of a spherical function we have

$$\int f(h_{i2} k'_2 a_2 k_2) dk'_2 dk_2 = f(h_{i2}) \omega_\nu(a_2) \int dk'_2 dk_2.$$

Thus

$$\begin{aligned} & \mathrm{Res}_{s_1=1} J(s_1, f, \phi) \\ &= 2 \left| \sum_{z \in \Lambda(D)} f(z) \right|^2 \int_{k' \in K} \int_{a>1} \int_K \phi(v_0 k' a k) \omega_\nu(a_2) \bar{\omega}_\nu(a_3) \delta(a) \frac{da}{a} dk' dk. \end{aligned}$$

To avoid evaluating the integral, we determine the asymptotic as $\tau \rightarrow \infty$, then

$$\begin{aligned} & \int_{k' \in K} \int_{a>1} \int_K \phi(v_0 k' a k) \omega_\nu(a_2) \bar{\omega}_\nu(a_3) \delta(a) \frac{da}{a} dk' dk, \text{ as } \tau \rightarrow \infty \\ & \sim \int_{k' \in K} \int_{a>1} \int_K \phi(v_0 k' a k) \delta(a) \frac{da}{a} dk' dk \end{aligned}$$

after change of variables $a \rightarrow e^u$ and $u \rightarrow \sinh^{-1}(w/\tau)$. Note that

$$\int_{k' \in K} \int_{a>1} \int_K \phi(v_0 k' a k) \delta(a) \frac{da}{a} dk' dk = \int_{G(\mathbb{R})} \phi(v_0 g) dg,$$

then the proof follows. \square

4.2. Shintani integral on the PVS of pairs of quaternary alternating 2-forms. In this section we construct a Shintani integral for the PVS of pairs of quaternary alternating 2-forms. As before, let $H(\mathbb{R})$, $B(\mathbb{R})$ be the subgroup of $B_2^+(\mathbb{R}) \times P_{2,2}^+(\mathbb{R})$, $B_2^+(\mathbb{R}) \times P_{\min}^+(\mathbb{R})$ defined by

$$\{g = (h_1, h_2) \mid \det(h_1) \det(h_2) = 1\},$$

where $B_2^+(\mathbb{R}) = B_2(\mathbb{R}) \cap \mathrm{GL}_2^+(\mathbb{R})$ the subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ and $P_{2,2}^+(\mathbb{R})$, $P_{\min}^+(\mathbb{R})$ the subgroup of $\mathrm{GL}_4^+(\mathbb{R})$ consisting of

$$\begin{pmatrix} a_3 & b_3 & s & t \\ c_3 & d_3 & u & v \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 & s & t \\ 0 & d_3 & u & v \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & 0 & d_2 \end{pmatrix}$$

with positive determinant of upper left 2×2 matrices respectively. Denote by $T(\mathbb{R})$ the subgroup acting on $W(\mathbb{R})$ trivially. We define $H^1(\mathbb{R})$ to be the

subgroup of $H(\mathbb{R})$ with the determinants of both upper-left and lower-right 2×2 matrices equal to 1. Note that

$$H(\mathbb{R})/T(\mathbb{R}) = \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} \times \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \cdot H^1(\mathbb{R}), t \in \mathbb{R}^+.$$

We have similar decomposition for $B(\mathbb{R})$,

$$B(\mathbb{R})/T(\mathbb{R}) = \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} \times \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) \cdot B^1(\mathbb{R}), t \in \mathbb{R}^+.$$

Finally denote a Levi subgroup of $B(\mathbb{R})$ by

$$M(\mathbb{R}) = \left(\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \times \begin{pmatrix} a_3 & b_3 & 0 & 0 \\ 0 & d_3 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & 0 & d_2 \end{pmatrix} \right),$$

as well as an unipotent subgroup $U(\mathbb{R})$ generated by

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \times \begin{pmatrix} \pm 1 & 0 & s & t \\ 0 & \pm 1 & u & v \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Recall that $H(\mathbb{R})$ acts right on the $W_D(\mathbb{R})$, the subset of alternating 2-forms with fixed discriminant D and with vanishing r_1 -element. Define the character χ_0 and χ_1 of $H(\mathbb{R})$ by

$$P_0(wg) = \chi_0(g)P_0(w), \text{ and } P_1(wg) = \chi_1(g)P_1(w),$$

where $P_0(w) = r_2(w)$ and $P_1(w) = -\text{Pfaff}(M_w)$. The discrete group $B(\mathbb{Z})$ acts right invariantly on the space $W'_D(\mathbb{Z})$, the subset of $W_D(\mathbb{Z})$ with vanishing a -element and with $|r_2| = 1$, $|c| = 1$. Lastly, we denote $W'_D(\mathbb{R})$ the subset of $W_D(\mathbb{R})$ with vanishing a -element.

From now on we will use the coordinates g_2, g_3 to denote the lower-right and upper-left 2 by 2 corner of the matrix in $P_{2,2}$ or P_{\min} respectively. The measure on $H(\mathbb{R})$ is defined as follows. The measure on $B_2^+(\mathbb{R})$ is the usual left invariant measure. The measure on $P_{2,2}^+(\mathbb{R})$ is given by $dp = dmdn$ if we decompose p as

$$p = mn = \begin{pmatrix} a_3 & b_3 & 0 & 0 \\ c_3 & d_3 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & s & t \\ 0 & 1 & u & v \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where dm is the left invariant measure.

Now we are going to define a distribution, still call it the Shintani integral, on the PVS $(B(\mathbb{R}), W'_D(\mathbb{R}))$ by

$$J(s_0, s_1, f, \phi) = \int_{B(\mathbb{Z}) \backslash B(\mathbb{R})/T(\mathbb{R})} \Phi(s_0, s_1, g) f(g_2) \bar{f}(g_3) dg.$$

We first need to define a Schwartz function ϕ on $W(\mathbb{R})$. For $w \in W(\mathbb{R})$, define

$$\begin{aligned} \phi(w) = & \exp^{-\pi\tau(a(w)^2+b(w)^2+c(w)^2+d(w)^2)} \exp^{-\pi\tau(e(w)^2+f(w)^2+g(w)^2+h(w)^2)} \\ & \cdot \exp^{-\pi(r_2(w)^2+l_1(w)^2+l_2(w)^2)}, \end{aligned}$$

where τ is again a positive real parameter.

As we did in the last section, to the Schwartz function ϕ on $W(\mathbb{R})$ chosen above, we associate a mixed Eisenstein series of the group $B(\mathbb{R})$ by taking

$$\Phi(s_0, s_1, g) = \sum_{w \in W'_D(\mathbb{Z})} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{|\chi_0(g)|^{s_0}}{|n|^{s_0}} |\chi_1(g)|^{s_1} \phi(wg),$$

where $s_0, s_1 \in \mathbb{C}$.

Consider the embedding:

$$V_D(\mathbb{Z}) \rightarrow W_D(\mathbb{Z})$$

defined by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & e & f \\ -1 & 0 & g & h \\ -e & -g & 0 & 0 \\ -f & -h & 0 & 0 \end{pmatrix} \right),$$

from now on, $\hat{V}_D(\mathbb{Z})$ stands for the image of this embedding.

Lemma 4.2. *With the meaning $\hat{V}_D(\mathbb{Z})$ shown above, the Shintani integral can be expressed as*

$$J(s_0, s_1, f, \phi) = \frac{1}{\pi^2 \cdot h(D)} \int_{H(\mathbb{Z}) \backslash H(\mathbb{R})/T(\mathbb{R})} \Psi(s_0, s_1, g) f(g_2) \bar{f}(g_3) dg,$$

where the mixed Eisenstein series Ψ is given by

$$\Psi(s_0, s_1, g) = \sum_{w \in \hat{V}_D(\mathbb{Z})} \sum_{\gamma \in U(\mathbb{Z})} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{|\chi_0(g)|^{s_0}}{|n|^{s_0}} |\chi_1(g)|^{s_1} \phi(w\gamma g).$$

Proof. The proof follows from the observation that different $1 \times \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ -orbit of $V_D(\mathbb{Z})$ in $\hat{V}_D(\mathbb{Z})$ are in the same orbit by the $U(\mathbb{Z})$ action. \square

We next compute its residue at some special values. The next proposition states the residue turns out to be exactly the square of sums over Heegner points again. The reason of the special values we choose here would be made clear in the main theorem of next section.

Theorem 4.3. *The Shintani integral $J(s_0, s_1, f, \phi)$ has a simple pole at $s_0 = 0$ and $s_1 = 1$ whose residue gives rise to the square product of periods over Heegner points. More precisely,*

$$- \frac{\pi^2 \Gamma(1) \zeta(2) h(D)}{4} \text{Res}_{s_0=0, s_1=1} J(s_0, s_1, f, \phi) \text{ as } \tau \rightarrow \infty \\ \sim L(\phi) \cdot \left| \sum_{z \in \Lambda_D}^* f(z) \right|^2.$$

Proof. Observing that

$$\int_{H(\mathbb{Z}) \backslash H(\mathbb{R})/T(\mathbb{R})} \Psi(s_0, s_1, g) f(g_2) \bar{f}(g_3) dg \\ = \sum_w \int_{y>0} \int_{H_w^1(\mathbb{Z}) \backslash H^1(\mathbb{R})} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{|\chi_0(g)|^{s_0}}{|n|^{s_0}} |\chi_1(g)|^{s_1} y^{s_0} \phi(wgy) f(g_2) \bar{f}(g_3) dg \frac{d^\times y}{y^4},$$

where w takes the images in $\hat{V}_D(\mathbb{Z})$ of representatives of $B_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})$ -orbit of $V_D(\mathbb{Z})$. Applying the classical Poisson summation formula, the integral has residue at $s_0 = 0$ the value

$$- 2 \int \sum_{w \in V_D(\mathbb{Z})} |\chi_1(g)|^{s_1} \phi(wg) f(g_2) \bar{f}(g_3) dg,$$

where the integral is over the group

$$(B_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})) \setminus (B_2^+(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})).$$

With the Eisenstein series, the above can be written as

$$- \frac{2}{\pi^{-(s_1-1)} \Gamma(s_1) \zeta(2s_1)} \int_{G(\mathbb{Z}) \backslash G(\mathbb{R})} \sum_{w \in V_D(\mathbb{Z})} E^*(s_1, g_1) \phi(wg) f(g_2) \bar{f}(g_3) dg.$$

The Eisenstein series has residue $\frac{1}{2}$ at $s_1 = 1$. Therefore, taking the residue at $s_1 = 1$, the above function becomes

$$\frac{-1}{\Gamma(1) \zeta(2)} \int_{G(\mathbb{Z}) \backslash G(\mathbb{R})} \sum_{w \in V_D(\mathbb{Z})} \phi(wg) f(g_2) \bar{f}(g_3) dg.$$

By the similar argument in the proof of theorem 4.1, as $\tau \rightarrow \infty$, the right side approximates to

$$4 \left| \sum_{z \in \Lambda_D}^* f(z) \right|^2 \cdot \frac{-1}{\Gamma(1) \zeta(2)} \int_{G(\mathbb{R})} \phi(v_0 g) dg,$$

where v_0 is an element in $V_D(\mathbb{R})$. □

By means of the functional equation of $E^*(s_1, g_1)$, we have

Corollary 4.4. *The residue of Shintani integral $J(s_0, s_1, f, \phi)$ at $s_0 = 0$ can be expressed by*

$$\text{Res}_{s_0=0} J(s_0, s_1, f, \phi) = \frac{\pi^{s_1} \Gamma(1 - s_1) \zeta(2(1 - s_1))}{\pi^{-(s_1-1)} \Gamma(s_1) \zeta(2s_1)} \text{Res}_{s_0=0} J(s_0, 1 - s_1, f, \phi).$$

5. CENTRAL VALUE OF GL_2 L-FUNCTION

We first define the group $H(\mathbb{Q})$ to be the subgroup of $B_2(\mathbb{Q}) \times P_{2,2}(\mathbb{Q})$ by the following condition

$$H(\mathbb{Q}) = \{(h_1, h_2) : \det(h_1) \det(h_2) = 1\}.$$

Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$. We then define the group $H(\mathbb{A})$ by the strong approximation

$$H(\mathbb{A}) = H(\mathbb{Q}) \cdot H(\mathbb{R}) \cdot O(\mathbb{R}) \cdot H(\mathbb{A}_f),$$

where $H(\mathbb{Q})$ is the diagonal embedding, $H(\mathbb{R})$ is defined in last section, $O(\mathbb{R})$ the subgroup $H(\mathbb{R}) \cap (O_2(\mathbb{R}) \times O_4(\mathbb{R}))$, and $H(\mathbb{A}_f)$ the subgroup of $\prod_p B_2(\mathbb{Z}_p) \times \prod_p P_{2,2}(\mathbb{Z}_p)$ with

$$H(\mathbb{Z}_p) = \{(h_1, h_2) : \det(h_1) \det(h_2) = 1\}.$$

Similarly, we can defined a Borel subgroup $B(\mathbb{A})$ of $H(\mathbb{A})$ which is inside $B_2(\mathbb{A}) \times P_{\min}(\mathbb{A})$.

Over \mathbb{A} , we will denote by $W(\mathbb{A})$ and its subset $W'(\mathbb{A})$ the set of pairs of quaternary alternating 2-forms with vanishing r_1 and r_1, a elements respectively. Clearly the space decomposes

$$W(\mathbb{A}) = W_{\infty} \times W(\mathbb{A}_f) = W(\mathbb{R}) \times W(\mathbb{A}_f).$$

As before, let $W_D(\mathbb{A})$ be the subset of $W(\mathbb{A})$ with

$$W_D(\mathbb{A}) = \{(w_{\infty}, w_{\mathbb{A}_f}) : \text{disc}(w_{\infty}) = D, \text{disc}(w_p) = D_p\}.$$

Then it is clear that $B(\mathbb{A})$ acts on $W'_D(\mathbb{A})$ invariantly. Last, denote by $T(\mathbb{A})$ the subgroup which acts trivially on $W(\mathbb{A})$. We will repeatedly identify

$$t \in Z(\mathbb{A})/T(\mathbb{A}) = \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} \times \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right), t \in \mathbb{A}^{\times}$$

in the following computations, where $Z(\mathbb{A})$ stands for the center of $H(\mathbb{A})$ and one of its components respectively.

For the Schwartz function on $W(\mathbb{A})$, we have $\mathcal{S}(W(\mathbb{A})) = \mathcal{S}(W(\mathbb{R})) \otimes \mathcal{S}(W(\mathbb{A}_f))$ and assume $\phi_{\mathbb{A}} \in \mathcal{S}(W(\mathbb{A}))$ decomposes into $\phi \otimes \phi_0$. We next fix ϕ_0 to be a kind of characteristic function of $\prod_p W(\mathbb{Z}_p)$ defined by

$$\begin{aligned} \phi_0(w_v) &= 1 \quad |r_2(w_v)|_v = |c(w_v)|_v = 1, w_v \in W(\mathbb{Z}_v) \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then it is easy to show that

Proposition 5.1. *Let*

$$\Phi_{\mathbb{A}}(s_0, s_1, g) = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \sum_{w \in W'_D(\mathbb{Q})} \frac{1}{|n|^{s_0}} |\chi_0(g)|^{s_0} |\chi_1(g)|^{s_1} \phi(wg) \phi_0(wg),$$

where $\phi(\cdot)$ is the Schwartz function of $W(\mathbb{R})$ defined in the last section. Then we have

$$J(s_0, s_1, f, \phi) = \int_{B(\mathbb{Q}) \backslash B(\mathbb{A})/T(\mathbb{A})} \Phi_{\mathbb{A}}(s_0, s_1, g) f(g_2) \bar{f}(g_3) dg.$$

5.1. Proof of Main Theorem 1.1. We have shown by theorem 4.3 that the residue of $J(s_0, s_1, f, \phi)$ at $s_0 = 0, s_1 = 1$ gives a product of the period integrals of the GL_2 Maass form f . Now by analyzing the local orbit integrals, we will express the same residue with the special value of the L-function of f .

Theorem 5.2. *On the right side of the last identity, we have the residue at $s_0 = 0, s_1 = 1$ and furthermore*

$$\begin{aligned} & \mathrm{Res}_{s_0=0, s_1=1} \int_{B(\mathbb{Q}) \backslash B(\mathbb{A})/T(\mathbb{A})} \Phi_{\mathbb{A}}(s_0, s_1, g) f(g_2) \bar{f}(g_3) dg \text{ as } \tau \rightarrow \infty \\ & \sim -\frac{L(\phi)}{2\pi^3 \Gamma(1) \zeta(2)} \|f\|^2 \frac{\zeta(2) L(1/2, \pi_{E_D})}{L(1, \pi, \mathrm{Ad}) L(1, \chi_D)} \prod_{v \in S} P_v, \end{aligned}$$

where E_D is the quadratic field extension over \mathbb{Q} with the fundamental discriminant D and S is the set of all ramified finite places.

Proof. In terms of the corollary 4.4, we are going to compute the

$$\int_{B(\mathbb{Q}) \backslash B(\mathbb{A})/T(\mathbb{A})} \Phi_{\mathbb{A}}(s_0, 1 - s_1, g) f(g_2) \bar{f}(g_3) dg.$$

Replacing the cusp form f by its Fourier-Whittaker expansion

$$f(g_3) = \sum_{\alpha \in \mathbb{Q}^\times} W_f \left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g_3 \right),$$

where

$$W_f \left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g_3 \right) = \int_{\mathbb{A}/\mathbb{Q}} f \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g_3 \right) \psi(-x) dx,$$

and unfolding the summation over $\alpha \in \mathbb{Q}^\times$, we have

$$\begin{aligned} & \int_{B(\mathbb{Q}) \backslash B(\mathbb{A})/T(\mathbb{A})} \Phi_{\mathbb{A}}(s_0, 1 - s_1, g) f(g_2) \bar{f}(g_3) dg \\ & = \int_{N(\mathbb{Q}) \backslash B(\mathbb{A})/T(\mathbb{A})} \Phi_{\mathbb{A}}(s_0, 1 - s_1, g) f(g_2) W_{\bar{f}}(g_3) dg \end{aligned}$$

where the element of $N(\mathbb{Q})$ is of the form

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \times \begin{pmatrix} 1 & b_3 & s & t \\ 0 & 1 & u & v \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & 0 & c_2 \end{pmatrix}.$$

Unfolding the sum over rational orbits in $W'_D(\mathbb{Q})$, the orbital integral becomes

$$\int_{N_w(\mathbb{Q}) \backslash B(\mathbb{A})/T(\mathbb{A})} \Phi_{\mathbb{A}}(s_0, 1 - s_1, g) f(g_2) W_{\overline{f}}(g_3) dg$$

where $w \in W'_D(\mathbb{Q})/N(\mathbb{Q})$, $N_w(\mathbb{Q})$ the stabilizer group in $N(\mathbb{Q})$. The key observation is that the action $N(\mathbb{Q})$ on $W'_D(\mathbb{Q})$ has a single orbit of the form

$$w = \left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & D/4 \\ 0 & 0 & -D/4 & 0 \end{pmatrix} \right).$$

Another key point is the stabilizer group of w in $B(\mathbb{A})$ is generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \pm \begin{pmatrix} 1 & b_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $b_3 \in \mathbb{A}$, so it is isomorphic to the unipotent subgroup $N_2(\mathbb{A})$ of $\mathrm{GL}_2(\mathbb{A})$. Therefore, integrating the rational stabilizer group and using Selberg-Rankin method, each orbit integral has an Euler factorization. Over the non-Archimedean local places, it is equal to

$$\prod_{p \neq \infty} \int W_f \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) W_{\overline{f}} \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ \cdot |y_1|^{2-2s_1} |y_2|_p^{1-s_1} |y_3|_p^{2-2s_1} |t|_p^{s_0} d^\times t d^\times y_1 du_1 d^\times y_2 du_2 \frac{d^\times y_3}{|y_3|_p}.$$

One should bear in mind that at every finite place the integral is over the set of $g \in N_w(\mathbb{Q}_p) \backslash B(\mathbb{Q}_p)$ such that wg is inside the support of Schwartz function ϕ_0 , and its computation reduces to the orbit counting: how many orbits in $W'_D(\mathbb{Z}_p)$ represented by

$$\left(\begin{pmatrix} 0 & 0 & 0 & y_1 y_3 \\ 0 & 0 & y_1 y_2 y_3 & d \\ 0 & -y_1 y_2 y_3 & 0 & 0 \\ -y_1 y_3 & -d & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t & e & f \\ -t & 0 & g & h \\ -e & -g & 0 & 0 \\ -f & -h & 0 & 0 \end{pmatrix} \right).$$

Let us ask the question: how many orbits in $W'_D(\mathbb{Z}_p)$ such that $t \in \mathbb{Z}_p^\times$, $y_1 y_2 y_3 \in \mathbb{Z}_p^\times$ and $y_1 y_3 \in \mathbb{Z}_p$. The answer is: the number of orbits is equal

to the number of solutions of x such that $x^2 = D \pmod{4|y_2|_p^{-1}}$. Therefore, after summing all orbit integrals we have at the finite place p

$$\int A(D, |y_2|_p^{-1}) W_f \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) W_{\bar{f}} \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot |y_2|_p^{1-s_1} |y_3|_p^{-1} d^\times y_2 d^\times y_3.$$

Therefore, the integral $J(1-s_0, s_1, f, \phi)$ can be factored at each place an Euler product. At all finite places, it is the product

$$(1) \quad \prod_{p \neq \infty} \int A(D, |y_2|_p^{-1}) W_f \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) W_{\bar{f}} \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot |y_2|_p^{1-s_1} d^\times y_2 \frac{d^\times y_3}{|y_3|_p}.$$

At the real Archimedean place, in the local orbit integral if we integrate the stabilizer group $\left(\begin{pmatrix} y_3^{-1} & 0 \\ 0 & y_3 \end{pmatrix} \times \begin{pmatrix} y_3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, y_3 \in \mathbb{R}^+ \right)$ first, the inner integral is

$$(2) \quad \int_{y_3 > 0} \sqrt{2\pi y_2 y_3} K_{\nu-1/2}(2\pi y_2 y_3) \cdot \sqrt{2\pi y_3} K_{\nu-1/2}(2\pi y_3) e^{2\pi i y_2 y_3 x_2} \frac{d^\times y_3}{y_3}.$$

Then the orbit integral contains the product of (1) and (2). By means of the functional equation of $E^*(s, g_3)$, it can be written as the product of

$$(3) \quad \prod_{p \neq \infty} \int A(D, |y_2|_p^{-1}) W_f \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) W_{\bar{f}} \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot |y_2|_p^{1-s_1} d^\times y_2 d^\times y_3$$

at all finite places and

$$(4) \quad \frac{\pi^{-(1-s-1)} \Gamma(1-s) \zeta(2(1-s))}{\pi^{-(s-1)} \Gamma(s) \zeta(2s)} \Big|_{s=0} \cdot \int_{y_3 > 0} \sqrt{2\pi y_2 y_3} K_{\nu-1/2}(2\pi y_2 y_3) \cdot \sqrt{2\pi y_3} K_{\nu-1/2}(2\pi y_3) e^{2\pi i y_2 y_3 x_2} d^\times y_3$$

at the real place.

The inner product

$$\int W_{\pi_p} \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) W_{\pi_p^\vee} \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) d^\times y$$

defines a $\mathrm{GL}_2(\mathbb{Q}_p)$ -invariant inner product, the non-Archimedean case is in [Ber84] and the Archimedean case is proved by Baruch [Bar03], which is

needed in our real place computation.

Therefore, the integral

$$\int W_f \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} g_2 \right) W_{\bar{f}} \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \right) d^\times y_3$$

is a spherical function, by the uniqueness of the spherical function, it is

$$C \cdot \sigma(g_2),$$

where by Macdonald formula the standard spherical function is

$$\sigma \left(\begin{pmatrix} \omega^n & 0 \\ 0 & 1 \end{pmatrix} \right) = \frac{p^{-n/2}}{1+p^{-1}} \left(\alpha^n \frac{1-\alpha^{-2}p^{-1}}{1-\alpha^{-2}} + \alpha^{-n} \frac{1-\alpha^2p^{-1}}{1-\alpha^2} \right),$$

where $\alpha = p^{-\lambda}$ and $\pi_p = \text{Ind}_{B_2}^{\text{GL}_2} (|\cdot|^\lambda \otimes |\cdot|^{-\lambda})$; and the constant C is given by

$$C = \int W_f \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \right) W_{\bar{f}} \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \right) d^\times y_3.$$

Therefore the orbit integral at all finite places is equal to

$$(5) \quad \prod_{p \neq \infty} \int_{y_3 \in \mathbb{Z}_p} W_f \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \right) W_{\bar{f}} \left(\begin{pmatrix} y_3 & 0 \\ 0 & 1 \end{pmatrix} \right) d^\times y_3 \\ \cdot \int_{y_2 \in \mathbb{Z}_p} A(D, |y_2|_p^{-1}) \sigma(y_2) |y_2|_p^{1-s_1} d^\times y_2.$$

The integral at the real place

$$\int_{y_3 > 0} \sqrt{2\pi y_2 y_3} K_{\nu-1/2}(2\pi y_2 y_3) \cdot \sqrt{2\pi y_3} K_{\nu-1/2}(2\pi y_3) e^{2\pi i y_2 y_3 x_2} d^\times y_3$$

is again a spherical function of $\text{SL}_2(\mathbb{R})$, it is equal to

$$(6) \quad \int_0^\infty K_{\nu-1/2}(y_3) K_{\nu-1/2}(y_3) d^\times y_3 \cdot \omega_\lambda(g_2).$$

Therefore the product of (3) and (4) becomes the product of

$$(7) \quad \prod_{p \neq \infty} \int_{y_2 \in \mathbb{Z}_p} A(D, |y_2|_p^{-1}) \sigma(y_2) |y_2|_p^{1-s_1} d^\times y_2,$$

and

$$(8) \quad - \frac{\|f\|^2}{2\pi} \omega_\lambda(g_2).$$

Note that the rest of the integral at the real place is

$$4 \int_{B_2(\mathbb{R})} \int_{t=0}^\infty \int_{x_1=-\infty}^\infty \int_{y_1=0}^\infty \int_{u \in U(\mathbb{R})} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \phi(w_0 t g) \omega_\lambda(g_2) \frac{t^{s_0}}{|n|^{s_0}} y^{1-s_1} dx_1 \frac{d^\times y_1}{y_1} dg_2 d^\times t du,$$

where $w_0 = \left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & D/4 \\ -1 & 0 & 0 & 0 \\ 0 & -D/4 & 0 & 0 \end{pmatrix} \right)$, g is the element of $B^1(\mathbb{R})$ with the given coordinates.

Now Poisson summation applies again to the inner summation. It follows that the residue of the orbit integral at $s_0 = 0$, as $\tau \rightarrow \infty$, approximates to

$$(9) \quad \prod_{p \neq \infty} \int_{y_2 \in \mathbb{Z}_p} A(D, |y_2|_p^{-1}) \sigma(y_2) |y_2|_p^{1-s_1} d^\times y_2 \\ \cdot \frac{2||f||^2}{\pi^3} \int_{g_2, g_3 \in \mathrm{SL}_2(\mathbb{R})} \int_{x_1 = -\infty}^{\infty} \int_{y_1 = 0}^{\infty} \phi(w'_0 g) y_1^{1-s_1} dx_1 \frac{d^\times y_1}{y_1} dg_2 dg_3,$$

where $w'_0 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & D/4 \end{pmatrix} \right)$, g is the element of $G(\mathbb{R})$ with the given coordinates.

At the place where p is split in the quadratic extension,

$$\int_{|y_2|_p \leq 1} A(D, |y_2|_p^{-1}) \sigma \left(\begin{pmatrix} y_2 & 0 \\ 0 & 1 \end{pmatrix} \right) d^\times y_2 \\ = (1 - p^{-1/2} \alpha)^{-1} (1 + p^{-1/2} \alpha) (1 + p^{-1})^{-1} \frac{1 - \alpha^{-2} p^{-1}}{1 - \alpha^{-2}} \\ + (1 - p^{-1/2} \alpha^{-1})^{-1} (1 + p^{-1/2} \alpha^{-1}) (1 + p^{-1})^{-1} \frac{1 - \alpha^2 p^{-1}}{1 - \alpha^2} \\ = \frac{(1 - p^{-1})(1 + \alpha p^{-1/2})(1 + \alpha^{-1} p^{-1/2})}{(1 + p^{-1})(1 - \alpha p^{-1/2})(1 - \alpha^{-1} p^{-1/2})} \\ = \frac{1}{1 + p^{-1}} \frac{L_p(1/2, \pi_{E_D})}{L_p(1, \pi, \mathrm{Ad})} = \frac{1 - p^{-1}}{1 - p^{-2}} \frac{L_p(1/2, \pi_{E_D})}{L_p(1, \pi, \mathrm{Ad})},$$

where $L_p(\cdot, \pi_{E_D})$ stands for the base change local L-function. At the place where p is unramified and nonsplit in the quadratic extension, we have

$$\int_{|y_2|_p \leq 1} A(D, |y_2|_p^{-1}) \sigma \left(\begin{pmatrix} y_2 & 0 \\ 0 & 1 \end{pmatrix} \right) d^\times y_2 \\ = (1 - p^{-1/2} \alpha)^{-1} (1 - p^{-1/2} \alpha) (1 + p^{-1})^{-1} \frac{1 - \alpha^{-2} p^{-1}}{1 - \alpha^{-2}} \\ + (1 - p^{-1/2} \alpha^{-1})^{-1} (1 - p^{-1/2} \alpha^{-1}) (1 + p^{-1})^{-1} \frac{1 - \alpha^2 p^{-1}}{1 - \alpha^2} \\ = 1 \\ = \frac{1}{1 - p^{-1}} \frac{L_p(1/2, \pi_{E_D})}{L_p(1, \pi, \mathrm{Ad})} = \frac{1 + p^{-1}}{1 - p^{-2}} \frac{L_p(1/2, \pi_{E_D})}{L_p(1, \pi, \mathrm{Ad})}.$$

Therefore the value at $s_1 = 1$ gives rise to

$$\begin{aligned}
& \prod_{p \neq \infty} \int_{y_2 \in \mathbb{Z}_p} A(D, |y_2|_p^{-1}) \sigma(y_2) d^\times y_2 \\
& \cdot \frac{2\|f\|^2}{\pi^3} \int_{g_2, g_3 \in \mathrm{SL}_2(\mathbb{R})} \int_{x_1=-\infty}^{\infty} \int_{y_1=0}^{\infty} \phi(w'_0 g) dx_1 \frac{d^\times y_1}{y_1} dg_2 dg_3 \\
& = \prod_{p \neq \infty} \int_{y_2 \in \mathbb{Z}_p} A(D, |y_2|_p^{-1}) \sigma(y_2) d^\times y_2 \cdot \frac{\|f\|^2}{\pi^4} \int_{g \in G(\mathbb{R})} \phi(w'_0 g) dg,
\end{aligned}$$

and our local computation shows that the Euler product at all unramified places is exactly the same of

$$\frac{\zeta(2)L(1/2, \pi_{E_D})}{L(1, \pi, \mathrm{Ad})L(1, \chi_D)},$$

so the proof follows. \square

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